Grade 7/8 Math Circles
October 18/19/20/24, 2022
Radians Solutions

Exercise Solutions

Exercise 1

Choose a whole number from 1 to 500 that you think would be a better choice for the circle to be split into. Find the number of positive divisors and compare that number with the 24 divisors of 360.

Exercise 1 Solution

Responses will vary. One choice is the number 500, which has 12 positive divisors: 1, 2, 4, 5, 10, 20, 25, 50, 100, 125, 250, 500. So, even though 500 is a greater number than 360, it still has half as many divisors.

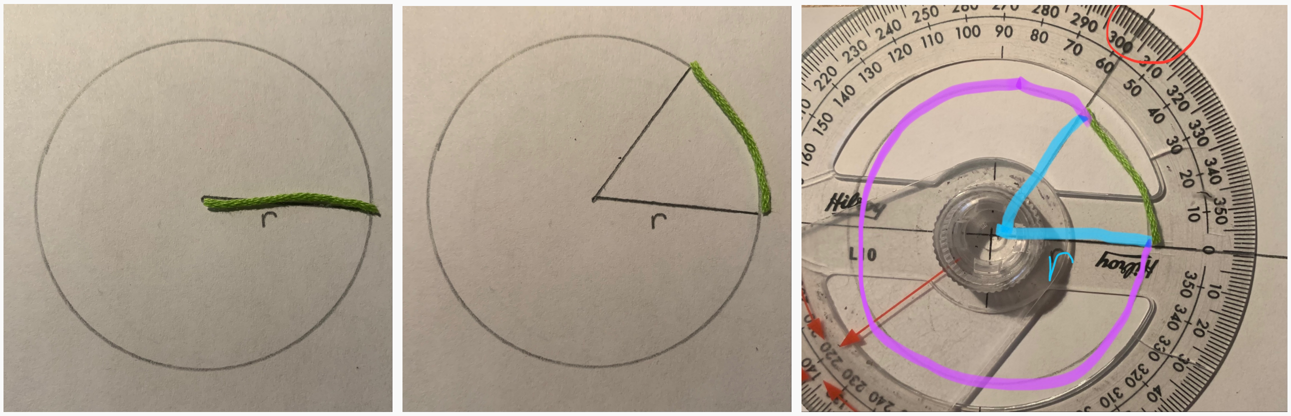
The number 360 has more divisors than any positive whole number smaller than 360 (so all positive whole numbers between 0 and 360). This means that 360 is highly composite.

Exercise 2

Draw a circle and a radius. Take a string and cut it to a length which is equal to the radius. Lay the string along the circumference of the circle and draw two radii from the centre of the circle to the points on the edge where the ends of the string are. Measure the central angle of the resulting minor sector.

Exercise 2 Solution

Is your central angle about 57°? This is a radian, a unit for measuring angles. Notice that we never chose a number to split the circle into.



Exercise 3

Convert the following to radians.

a) 3° b) 45° c) 90° d) 341° e) 360° f) 450°

Exercise 3 Solution

a) $3^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{60}$ rad

b) $45^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{4}$ rad

c) $90^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{2}$ rad

d) $341^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{341\pi}{180}$ rad

e) $360^\circ \times \frac{\pi \text{ rad}}{180^\circ} = 2\pi$ rad

f) $450^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{5\pi}{2}$ rad

Exercise 4

Convert the following to degrees.

a) $\frac{\pi}{30}$ rad b) $\frac{\pi}{7}$ rad c) π rad d) $\frac{3\pi}{2}$ rad e) 3π rad f) $\frac{7\pi}{9}$ rad

Exercise 4 Solution

a) $\frac{\pi}{30} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{180}{30}\right)^\circ = 6^\circ$

b) $\frac{\pi}{7} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{180}{7}\right)^\circ \approx 25.71^\circ$

c) $\pi \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = 180^\circ$

d) $\frac{3\pi}{2} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{3\times 180}{2}\right)^\circ = 270^\circ$

e) $3\pi \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = (3 \times 180)^\circ = 540^\circ$

f) $\frac{7\pi}{9} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{7\times 180}{9}\right)^\circ = 140^\circ$

Exercise 5

Calculate the arc length of sectors with the following central angles and radii.

a) radius = 52 cm, central angle = 2π rad

b) radius = 97 mm, central angle = 1 rad

c) radius = 4 m, central angle = $\frac{17\pi}{8}$ rad

d) radius = 8 mm, central angle = $\frac{9\pi}{25}$ rad

Exercise 5 Solution

a) $\text{arc length} = 52 \times 2\pi = 104\pi \text{ cm}$

b) $\text{arc length} = 97 \times 1 = 97 \text{ mm}$

c) $\text{arc length} = 4 \times \dfrac{17\pi}{8} = \dfrac{17\pi}{2} \text{ m}$

d) $\text{arc length} = 8 \times \dfrac{9\pi}{25} = \dfrac{72\pi}{25} \text{ mm}$

Exercise 6

Calculate the area of sectors with the following central angles and radii.

a) radius = 12 cm, central angle = 2π rad

b) radius = 6 mm, central angle = 4 rad

c) radius = 2 m, central angle = $\frac{5\pi}{3}$ rad

d) radius = 9 mm, central angle = $\frac{4\pi}{5}$ rad

Exercise 6 Solution

a) $$\begin{aligned} A_S &= \frac{1}{2} \times 12^2 \times 2\pi \\ &= \frac{1}{2} \times 144 \times 2\pi \\ &= 144\pi \text{ cm}^2 \end{aligned}$$

b) $$\begin{aligned} A_S &= \frac{1}{2} \times 6^2 \times 4 \\ &= \frac{1}{2} \times 36 \times 4 \\ &= 72 \text{ mm}^2 \end{aligned}$$

c) $$\begin{aligned} A_S &= \frac{1}{2} \times 2^2 \times \frac{5\pi}{3} \\ &= \frac{1}{2} \times 4 \times \frac{5\pi}{3} \\ &= \frac{10\pi}{3} \text{ m}^2 \end{aligned}$$

d) $$\begin{aligned} A_S &= \frac{1}{2} \times 9^2 \times \frac{4\pi}{5} \\ &= \frac{1}{2} \times 81 \times \frac{4\pi}{5} \\ &= \frac{162\pi}{5} \text{ mm}^2 \end{aligned}$$

Exercise 7

What are the following angles in degrees and in radians?

a) 200 gradians

b) 100 gradians

Exercise 7 Solution

a) $\frac{200}{400} = \frac{1}{2}$, so 200 gradians is half of a circle and 200 gradians $=$ 180 degrees $= \pi$ radians

b) $\frac{100}{400} = \frac{1}{4}$, so 100 gradians is a quarter of a circle and 100 gradians $=$ 90 degrees $= 2\pi$ radians

Problem Set Solutions

1. Convert the following to radians.

   a) 1° b) π° c) 15° d) 30° e) 120° f) 248° g) 613° h) 840°

Problem 1 Solution

a) $1^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{180}$ rad

b) $\pi^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi^2}{180}$ rad

c) $15^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{5\pi}{60}$ rad

d) $30^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{6}$ rad

e) $120^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{2\pi}{3}$ rad

f) $248^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{62\pi}{45}$ rad

g) $613^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{613\pi}{180}$ rad

h) $840^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{14\pi}{3}$ rad

2. Convert the following to degrees.

   a) 1 rad b) $\frac{\pi}{2}$ rad c) $\frac{2\pi}{5}$ rad d) 26 rad e) 4π rad f) $\frac{29\pi}{11}$ rad g) $\frac{\pi^2}{2}$ rad

Problem 2 Solution

a) $1 \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{180}{\pi}\right)^\circ \approx 57.296^\circ$

b) $\frac{\pi}{2} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{180}{2}\right)^\circ = 90^\circ$

c) $\frac{2\pi}{5} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{2\times180}{5}\right)^\circ = 72^\circ$

d) $26 \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{26\times180}{\pi}\right)^\circ \approx 1489.690^\circ$

e) $4\pi \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = (4 \times 180)^\circ = 720^\circ$

f) $\frac{29\pi}{11} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{29\times180}{11}\right)^\circ \approx 474.545^\circ$

g) $\frac{\pi^2}{2} \text{ rad} \times \frac{180^\circ}{\pi \text{ rad}} = \left(\frac{180\pi}{2}\right)^\circ \approx 282.743^\circ$

3. To convert from degrees to radians, we multiply an angle measured in degrees by $\frac{\pi \text{ rad}}{180^\circ}$ to cancel out the degree units. To convert from radians to degrees, we multiply an angle measured in radians by $\frac{180^\circ}{\pi \text{ rad}}$ to cancel out the radian units. In both cases, we are multiplying by one since $180^\circ = \pi$ rad.

   Using the idea of multiplying by one and cancelling out the given units, find the factor by which we multiply an angle measured in gradians to convert into degrees and the factor by which we multiply an angle measured in gradians to convert into radians.

Problem 3 Solution

From the lesson, we know that 400 gradians is equal to 360°. So, $\frac{360^\circ}{400 \text{ gradians}} = \frac{9^\circ}{10 \text{ gradians}} = 1$. That means, the factor we'd use to convert from gradians to degrees would be $\frac{9^\circ}{10 \text{ gradians}}$.

From the lesson, we know that 400 gradians is equal to 2π rad. So, $\frac{2\pi \text{ rad}}{400 \text{ gradians}} = \frac{\pi \text{ rad}}{200 \text{ gradians}} = 1$. That means, the factor we'd use to convert from gradians to radians would be $\frac{\pi \text{ rad}}{200 \text{ gradians}}$.

4. Calculate the circumference of a circle whose radius is 17 mm.

Problem 4 Solution

Using the formula from the lesson, $C = 2 \times \pi \times 17$

$$= 34\pi \text{ mm}$$

So, the circle has a circumference of 34π mm.

5. Calculate the area of a circle whose diameter is 6 cm.

Problem 5 Solution

From the lesson, we have the formula $A = \pi r^2$ where A represents the area of the circle and r represents the radius of the circle. Since we were given the diameter, we need to figure out the radius. We know that the radius is half of the diameter, and since the diameter is 6 cm, we know that the radius is 3 cm. We can now use the formula to find the area.

$$\begin{aligned} A &= \pi \times 3^2 \\ &= 9\pi \end{aligned}$$

So, the circle has an area of 9π cm^2.

6. Calculate the arc length and sector area of sectors with the following central angles and radii.

   a) radius $= 2$ mm, central angle $= 6$ rad

   b) radius $= 3$ cm, central angle $= \frac{\pi}{11}$ rad

   c) radius $= 7$ mm, central angle $= 0$ rad

   d) radius $= 1$ m, central angle $= \frac{87\pi}{46}$ rad

Problem 6 Solution

a)
$$\begin{aligned}\text{arc length} &= 2 \times 6 \\ &= 12 \text{ mm}\end{aligned} \qquad \begin{aligned} A_S &= \frac{1}{2} \times 2^2 \times 6 \\ &= \frac{1}{2} \times 4 \times 6 \\ &= 12 \text{ mm}^2\end{aligned}$$

b)
$$\begin{aligned}\text{arc length} &= 3 \times \frac{\pi}{11} \\ &= \frac{3\pi}{11} \text{ cm}\end{aligned} \qquad \begin{aligned} A_S &= \frac{1}{2} \times 3^2 \times \frac{\pi}{11} \\ &= \frac{1}{2} \times 9 \times \frac{\pi}{11} \\ &= \frac{9\pi}{22} \text{ cm}^2\end{aligned}$$

c)
$$\begin{aligned}\text{arc length} &= 7 \times 0 \\ &= 0 \text{ mm}\end{aligned} \qquad \begin{aligned} A_S &= \frac{1}{2} \times 7^2 \times 0 \\ &= \frac{1}{2} \times 14 \times 0 \\ &= 0 \text{ mm}^2\end{aligned}$$

d)
$$\begin{aligned}\text{arc length} &= 1 \times \frac{87\pi}{46} \\ &= \frac{87\pi}{46} \text{ m}\end{aligned} \qquad \begin{aligned} A_S &= \frac{1}{2} \times 1^2 \times \frac{87\pi}{46} \\ &= \frac{1}{2} \times 1 \times \frac{87\pi}{46} \\ &= \frac{87\pi}{92} \text{ m}^2\end{aligned}$$

7. Consider a circle whose radius is 3 mm that contains a sector with a central angle of $\frac{10\pi}{11}$ rad. Calculate the arc length of the **major sector**.

Problem 7 Solution

Since $\frac{10\pi}{11} < \pi$, $\frac{10\pi}{11}$ is the central angle for the minor sector, but we want to find the arc length of the major sector. There are 2 ways to go about this question.

First method: find the other angle

The simplest method is to recognize that a full circle has the angle 2π rad and that the major sector will have a central angle of $2\pi - \frac{10\pi}{11} = \frac{12\pi}{11}$ rad. Now, we can use our formula to find the arc length.

$$\begin{aligned}\text{arc length} &= 3 \times \frac{12\pi}{11} \\ &= \frac{36\pi}{11}\end{aligned}$$

So, the arc length of the major sector is $\frac{36\pi}{11}$ mm.

Second method: find the circumference

Another method is to find the arc length of the minor sector, then the circumference, and then find the arc length of the major sector.

$$\begin{aligned}\text{arc length}_{(\text{minor sector})} &= 3 \times \frac{10\pi}{11} \\ &= \frac{30\pi}{11}\end{aligned} \qquad \begin{aligned}C &= 2 \times \pi \times 3 \\ &= 6\pi\end{aligned}$$

Since the arc length of the minor sector is $\frac{30\pi}{11}$ mm and the total circumference is 6π mm, the arc length of the major sector is $6\pi - \frac{30\pi}{11} = \frac{36\pi}{11}$ mm.

8. Consider a circle whose radius is 5 mm that contains a sector with a central angle of $\frac{7\pi}{4}$ rad. Calculate the sector area of the **minor sector**.

Problem 8 Solution

Since $\pi < \frac{7\pi}{4} < 2\pi$, $\frac{7\pi}{4}$ is the central angle for the major sector, but we want to find the sector area of the minor sector. There are 2 ways to go about this question.

First method: find the other angle

The simplest method is to recognize that a full circle has the angle 2π rad and that the minor sector will have a central angle of $2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$ rad. Now, we can use our formula to find the sector area.

$$\begin{aligned} A_S &= \frac{1}{2} \times 5^2 \times \frac{\pi}{4} \\ &= \frac{1}{2} \times 25 \times \frac{\pi}{4} \\ &= \frac{25\pi}{8} \end{aligned}$$

So, the arc length of the minor sector is $\frac{25\pi}{8}$ mm^2.

Second method: find the total area

Another method is to find the sector area of the major sector, then the total area, and then find the sector area of the minor sector.

$$\begin{aligned} A_{S(\text{major sector})} &= \frac{1}{2} \times 5^2 \times \frac{7\pi}{4} \\ &= \frac{1}{2} \times 25 \times \frac{7\pi}{4} \\ &= \frac{175\pi}{8} \end{aligned} \qquad \begin{aligned} A &= \pi \times 5^2 \\ &= 25\pi \end{aligned}$$

Since the sector area of the major sector is $\frac{175\pi}{8}$ mm^2 and the total area is 25π mm^2, the sector area of the minor sector is $25\pi - \frac{175\pi}{8} = \frac{25\pi}{8}$ mm^2.

9. Consider a circle whose radius is 7 cm. Calculate the perimeter of the sector with a central angle of $\frac{3\pi}{2}$ rad.

   Hint: the perimeter of the sector is more than just the arc length. Review the definition of a sector from the lesson.

Problem 9 Solution

The perimeter of a sector is the sum of 2 radii and the arc length. So, we will first use our formula to find the arc length.

$$\begin{aligned}\text{arc length} &= 7 \times \frac{3\pi}{2} \\ &= \frac{21\pi}{2}\end{aligned}$$

Now, we take the sum of two radii and the arc length: $7 + 7 + \frac{21\pi}{2} = 14 + \frac{21\pi}{2}$

$$= \frac{28 + 21\pi}{2}$$

Therefore, the perimeter of the sector is $\frac{28+21\pi}{2}$ cm.

10. In the lesson, we rearranged the formula $\theta = \frac{a}{r}$ to get the arc length formula $a = r \times \theta$.

    We can also rearrange the sector area formula $A_S = \frac{1}{2}r^2\theta$. For example, suppose we know the sector area and the radius. If we want to calculate the central angle of the sector, we can rearrange the equation to isolate θ by doing the same operations on both sides of the equation.

    - First, multiply both sides of the equation by 2: $2A_S = r^2\theta$
    - Then, divide both sides of the equation by r^2: $\frac{2A_S}{r^2} = \theta$

    So, we get the following formula for calculating the central angle: $\theta = \frac{2A_S}{r^2}$.

    Rearrange the sector area formula in the same way but isolate the radius. That is, we know the sector area and the central angle but want to calculate the radius.

Problem 10 Solution

- First, multiply both sides of the equation by 2: $2A_S = r^2\theta$
- Then, divide both sides of the equation by θ: $\frac{2A_S}{\theta} = r^2$
- Then, take the square root of both sides of the equation: $\sqrt{\frac{2A_S}{\theta}} = r$, since $r > 0$

So, we get the following formula for calculating the radius: $r = \sqrt{\frac{2A_S}{\theta}}$.